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PIECEWISE CONSTANT HIGH-GAIN ADAPTIVE λ-TRACKING FOR HIGHER RELATIVE DEGREE LINEAR SYSTEMS[1]

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Abstract: In this paper we present an adaptive high-gain observer-based controller for linear, minimum-phase systems with known relative degree larger than one and known sign of the high-frequency gain. The adaptation scheme does not use any identification mechanism and is universal in the sense that it is independent of the system the controller is designed for. We prove that for any bounded and sufficiently smooth reference signal the modulus of the tracking error becomes smaller than an arbitrary small but fixed error bound λ if t tends to infinity. All states of the nonlinear closed-loop system remain bounded. *Copyright © 1999 IFAC*

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1. INTRODUCTION

Adaptive control is a popular method for robustly controlling systems with unknown parameters. One approach is to identify the parameters of the systems and to use this information for tuning the controller, (see for example (Åström and Wittenmark, 1995; Narendra, 1991) for a survey). In *non-identifier-based adaptive control*, as treated in this paper, the controller is directly tuned without estimating the parameters of the plant (see (Ilchmann, 1991) for a survey).

In this note we solve the adaptive λ-tracking problem of single-input, single-output minimum phase systems with known relative degree larger than one. λ-tracking means that the reference signal is not tracked asymptotically but within an arbitrarily small but prespecified ball of radius $\lambda > 0$. Within the setup of adaptive control which does not use any identification mechanisms or sufficiently high probing signals, several authors have tackled high-gain adaptive stabilization or tracking of minimum phase systems of higher relative degree, see (Khalil and Saberi, 1987; Mareels, 1984; Mårtensson, 1986; Miller and Davison, 1991; Morse, 1987; Mudgett and Morse, 1985; Mudgett and Morse, 1989). The contributions (Miller and Davison, 1991) and (Mudgett and Morse, 1989) are close to our approach. In (Mudgett and Morse, 1989) a similar observer-based feedback law is used. However, as they say in their Concluding Remarks, "the price being paid is a restriction on the control objectives that can be realized". In the present note we overcome this difficulty by introducing a dead-zone in the gain adaptation as in (Miller and Davison, 1991). The two main advantages of the λ-tracking approach are that, first, the controller does not incorporate any internal or reference model of the system and works for a large class of reference signals and, secondly, the controller is robust against noise corrupting the output. Although there is an overlap with the above mentioned contribution, we try to carry

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over the λ-tracking approach since it also has been successfully applied to nonlinear systems. The latter, not dealt with in the present note, is what we are heading for.

In this paper an observer-based non-identifier-based adaptive controller is proposed for linear minimum phase systems with arbitrary but known relative degree larger than one and positive high-frequency gain. Neither the dimension of the system nor any bounds on its parameters, besides a lower bound on the high-frequency gain, are required. This makes the controller very robust. The observer is an adaptive version of (Nicosia and Tornambè, 1989) as in (Bullinger and Allgöwer, 1997) with dimension equal to the relative degree of the system. The controller is an observer-state feedback whose coefficients depend on the same gain as the observer. In contrary to e.g. (Mareels, 1984) a lower bound on the high-frequency gain is needed, not an upper one. This controller can be used for tracking sufficiently smooth bounded signals.

The paper is organized as follows. After stating the system class and explaining the structure of the controller in Section 2, the results with the outline of the proof are presented in Section 3. The paper finishes with conclusions in Section 4.

2. PRELIMINARIES

System class

We consider linear systems with known relative degree r, having one input and one output and stable zero-dynamics, therefore being stabilizable and detectable. They can be described by the differential equations

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad A \in \mathbb{R}^{n\times n}, \tag{1a}$$

$$y(t) = \boldsymbol{c}\boldsymbol{x}(t), \qquad \boldsymbol{x}(t), \boldsymbol{b}, \boldsymbol{c}^T \in \mathbb{R}^n \tag{1b}$$

with

$$cA^i b = 0 \qquad \text{for } i = 0, \dots, r-2 \tag{2a}$$

$$cA^{r-1}b = g \geq \bar{g} > 0, \tag{2b}$$

$$\det \begin{bmatrix} sI_n - A & b \\ c & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}}_+ \tag{2c}$$

where (2a), (2b) are the relative degree conditions and (2c) guarantees minimum phaseness.

Objective

The control objective is to track a reference signal $y_{ref}(.) \in W^{r,\infty}$. $W^{r,\infty}$ is the set of all bounded *functions that are absolutely continuous on com*pact subintervals and whose r first derivatives are essentially bounded.

For this an adaptive output-feedback controller is designed in the state-space. It consists of an adaptive high-gain observer and an adaptive high-gain controller, both described in the following.

Observer

The observer is an adaptive version of the high-gain observer introduced by Nicosia and Tornambè (Nicosia and Tornambè, 1989) as in (Bullinger and Allgöwer, 1997). A state-space representation is

$$\dot{\hat{\boldsymbol{x}}}(t) = \hat{A}_\kappa \hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{b}}_\kappa e(t) \tag{3a}$$

$$e(t) = y(t) - y_{ref}(t) \tag{3b}$$

with

$$\hat{A}_{\kappa} = \begin{bmatrix} -p_{r-1}\cdot\kappa & 1 & 0 & & \\ -p_{r-2}\cdot\kappa^2 & 0 & 1 & & \\ \vdots & & & \ddots & \\ -p_1\cdot\kappa^{r-1} & 0 & 0 & & 1 \\ -p_0\cdot\kappa^r & 0 & 0 & & 0 \end{bmatrix}, \quad \hat{\boldsymbol{b}}_\kappa = \begin{bmatrix} p_{r-1}\cdot\kappa \\ p_{r-2}\cdot\kappa^2 \\ \vdots \\ p_1\cdot\kappa^{r-1} \\ p_0\cdot\kappa^r \end{bmatrix}.$$

Note that if the parameters p_i are chosen such that $s^r + \sum_{i=0}^{r-1} p_i s^i$ is a Hurwitz polynomial, then for any positive value of the observer gain κ, the spectrum of $\hat{A}_\kappa$ lies in the open left half plane, $\sigma(\hat{A}_\kappa) \subset \mathbb{C}_-$ and the observer dynamics are stable. The piecewise constant high-gain parameter κ is adapted according to the adaptation law described below.

Controller

The controller is an observer-state feedback

$$u = -\boldsymbol{q}_k \hat{\boldsymbol{x}}, \tag{4}$$

where

$$\boldsymbol{q}_k = \begin{bmatrix} q_0\cdot k^r, & \cdots, & q_{r-1}\cdot k \end{bmatrix}. \tag{5}$$

The motivation for such a controller with controller gain k comes from the fact that the transfer function from y to $\hat{\boldsymbol{x}}_i$ (the i-th state of the observer)

$$\frac{\hat{\boldsymbol{x}}_i(s)}{y(s)} = s^{i-1}\left(1 - \frac{\left(\frac{s}{\kappa}\right)^n + \ldots + \left(\frac{s}{\kappa}\right)^{n-i+1} p_{i-1}}{\left(\frac{s}{\kappa}\right)^n + \left(\frac{s}{\kappa}\right)^{n-1} p_1 + \ldots + p_n}\right)$$

is at "low" frequencies, where $\left(\frac{s}{\kappa}\right)$ is small, approximately a series a differentiators, i.e.

$$\hat{\boldsymbol{x}}_i(s) \approx s^{i-1} y(s).$$

The observer states approximate the derivatives of the output y with a band-width proportional to the observer gain κ. Therefore, the controller is approximately a $PD...D^{r-1}$ controller, i.e.

$$u(s) \approx -k^r \sum_{i=0}^{r-1} q_i \left(\frac{s}{k}\right)^i y(s). \tag{6}$$

As will be shown in Section 3, this allows to trea+ the higher relative degree case.

Gain Adaptation

The gain adaptation is performed in the following way: As long as the amplitude of the tracking error $e(t) = y(t) - y_{ref}(t)$ is larger than a user-defined bound $\lambda > 0$, i.e. the tracking error is outside the λ-strip, a gain φ is increased:

$$\dot{\varphi}(t) = d_\lambda(e(t))^2, \qquad \varphi(0) = k_0 \tag{7a}$$

$$d_\lambda(e) = \begin{cases} |e| - \lambda & \text{for } |e| \geq \lambda, \\ 0 & \text{for } |e| \leq \lambda. \end{cases} \tag{7b}$$

The controller gain $k(\cdot)$ takes only discrete values k_i:

$$k(t) = k_i \quad \text{if } \varphi(t) \in [k_i, k_{i+1}) \tag{7c}$$

where $\{k_i\}_{i\in\mathbb{N}_0}$ is a given strictly increasing sequence such that $k_0 > 0$ and for each $\gamma > 1$ there exists an $i^* \in \mathbb{N}$ such that

$$k_{i+1} - k_i > \gamma^i k_i^{2(r-1)} \quad \text{for all } i \geq i^*. \tag{7d}$$

The observer gain also takes discrete values and has to grow sufficiently faster than the controller gain. For simplicity, we choose $\kappa_i = k_i^2$ for all $i \geq 0$ in this paper, but other choices satisfying the growth condition work as well.

3. RESULTS

We will now show that the controller consisting of the observer (3), feedback (4) and adaptation (7) achieves λ-tracking for the large class of reference signals considered.

Theorem 1 *If $s^r + \sum_{i=0}^{r-1} p_i s^i$ and $s^r + \gamma \sum_{i=0}^{r-1} q_i s^i$ are Hurwitz polynomials for all $\gamma \geq \bar{g}$, then the application of the λ-tracker (3), (4) and (7) with $\kappa_i = k_i^2$ for all $i \geq 0$ to any system of the class (1), (2) and to any reference signal $y_{ref}(\cdot) \in W^{r,\infty}$ results in a closed-loop system which, independently of the initial values $x(0) \in \mathbb{R}^n$, $\hat{x}(0) \in \mathbb{R}^r$, $k_0 > 0$ has a unique solution*

$$(x, \hat{x}, \varphi) : [0, \infty) \to \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^+$$

which exists on the whole positive half axis and, moreover,

a) $(x(\cdot), \hat{x}(\cdot), \varphi(\cdot)) \in L_\infty(0, \infty)$,
b) there exists a $t^* > 0$ *such that* $k(t) = k(t^*)$ *for all* $t \geq t^*$,
c) $\lim_{t\to\infty} dist(|y(t) - y_{ref}(t)|, [0, \lambda]) = 0$.

Theorem 1 states that for any linear system with known relative degree and lower bound of the high-frequency gain a λ-tracking controller can be designed with the guarantee that all states and adaptation parameters remain bounded and that the tracking error $y - y_{ref}$ asymptotically converges to the λ-strip. The width of this strip is a parameter which can be chosen by the user and will usually depend on the specifications, on model uncertainties and on the measurement quality.

The following lemma will be used in the proof of Theorem 1.

Lemma 2 (Normal form) *Every minimum-phase single-input single-output linear system $(\tilde{A}, \tilde{b}, \tilde{c})$ of order n with relative degree r is similar to* $\left[\begin{array}{c|c} A & b \\ \hline c & 0 \end{array}\right] =$

$$\left[\begin{array}{cccccccc|c}
0 & 1 & & & & & & & 0 \\
\vdots & & \ddots & & & & & & \vdots \\
0 & & & 1 & & & & & 0 \\
-\alpha_0 & \cdots & & -\alpha_{r-1} & \cdots & & & -\alpha_{n-1} & g \\
 & & & & \tilde{A}_{22} & & & & 0 \\
 & & & & & & & & \vdots \\
 & & & & * & \cdots & * & & \vdots \\
 & & & & \vdots & \ddots & \vdots & \tilde{A}_{33} & \vdots \\
1 & & & & * & \cdots & * & & 0 \\
\hline
1 & 0 & \cdots & & & & & 0 & 0
\end{array}\right] \tag{8}$$

where $\sigma(\tilde{A}_{22}) \subset \mathbb{C}_-$, $\sigma(\tilde{A}_{33}) \subset \mathbb{C}_-$ and the stars indicate for real entries. All other entries are zero.

Proof of Lemma 2 *Any stabilizable linear system can be represented by*

$$\dot{\tilde{x}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \tilde{x} + \begin{bmatrix} \tilde{b}_1 \\ 0 \end{bmatrix} u, \quad \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$
$$y = \tilde{c}\tilde{x} = \begin{bmatrix} \tilde{c}_1 & \tilde{c}_2 \end{bmatrix} \tilde{x} \tag{9}$$

where $(\tilde{A}_{11}, \tilde{b}_1, \tilde{c}_1) \in \mathbb{R}^{n_1 \times n_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ is in controllability normal form, i.e.

$$\left[\begin{array}{cccccc|c}
0 & 1 & & & & & 0 \\
\vdots & & \ddots & \ddots & & & \vdots \\
0 & \cdots & & & 0 & 1 & \vdots \\
-a_0 & \cdots & & & & -a_{n_1-1} & g \\
\hline
c_0 & \cdots & c_{n_1-r-1} & 1 & 0 \cdots & 0 & 0
\end{array}\right]$$

where $\tilde{A}_{22} \in \mathbb{R}^{n_2 \times n_2}$ has $\sigma(\tilde{A}_{22}) \subset \mathbb{C}_-$. A straightforward computation shows that a possible transformation for mapping (9) into the normal form (8) is given by

$$T = \begin{bmatrix} & \tilde{c} & \\ & \tilde{c}\tilde{A} & \\ & \vdots & \\ & \tilde{c}\tilde{A}^{r-1} & \\ 0 & 0 & I_{n_2} \\ I_{n_1-r} & 0 & 0 \end{bmatrix}.$$

■

Proof of Theorem 1 *We will prove Theorem 1 in five steps. First, it is proven that the solution of the differential equations exists for all times and thus there is no finite escape time. In the second step we prove via contradiction that for sufficiently large i the Lyapunov function decreases exponentially on each time interval $[t_i, t_{i+1})$ and therefore the adaptation parameter φ converges as does the controller gain k. Then, boundednes* *of the observer states $\hat{\boldsymbol{x}}$ and thus of the plant input u as well as boundedness of the plant states $\boldsymbol{x}$ is shown in steps 3 and 4. The proof concludes by showing that the tracking error converges to the λ-strip.*

1) Global existence of a unique solution.
By Lemma 2, we may assume that the system is given in the normal form (8). The nonlinear closed-loop system (1), (3), (4) and (7) is of the form

$$\begin{aligned}
\dot{\boldsymbol{x}} &= A\boldsymbol{x} - \boldsymbol{b}\boldsymbol{q}_k\hat{\boldsymbol{x}}, & \boldsymbol{x}(0) &= \boldsymbol{x}_0 \in \mathbb{R}^n & \text{(10a)}\\
\dot{\hat{\boldsymbol{x}}} &= \hat{A}_\kappa\hat{\boldsymbol{x}} + \hat{\boldsymbol{b}}_\kappa e, & \hat{\boldsymbol{x}}(0) &= \hat{\boldsymbol{x}}_0 \in \mathbb{R}^r & \text{(10b)}\\
\dot{\varphi} &= d_\lambda(e)^2, & \varphi(0) &= k_0 > 0, & \text{(10c)}\\
e &= \boldsymbol{c}\boldsymbol{x} - y_{ref}. & & & \text{(10d)}
\end{aligned}$$

Since $t \mapsto \varphi(t)$ is non-decreasing and $t \mapsto k(t)$ is piecewise constant, it follows from the theory of ordinary differential equations that the solution of (10) is unique and exists on the whole of $[0,\infty)$.

2) Boundedness of the adaptation parameters.
Using the coordinates

$$\bar{\boldsymbol{x}} = \left[\boldsymbol{\xi}^T,\ \boldsymbol{z}^T,\ \hat{\boldsymbol{e}}^T\right]^T \tag{11}$$

where $\boldsymbol{\xi}^T = \left[x_1 - y_{ref},\ x_2 - \dot{y}_{ref},\ \ldots,\ x_r - y_{ref}^{(r-1)}\right]$ denotes the tracking error and its derivatives, $\boldsymbol{z} = \left[x_{r+1},\ \ldots,\ x_n\right]$ are the uncontrollable states ($\tilde{\boldsymbol{x}}_2$ in (9)) and those of the zero-dynamics, and $\boldsymbol{e} = [x_1,\ldots,x_r]^T - \hat{\boldsymbol{x}}$ is the observer error, the closed-loop system can be described by

$$\dot{\bar{\boldsymbol{x}}} = \bar{A}\bar{\boldsymbol{x}} - \bar{B}\boldsymbol{y}_{ref}$$
$$y = \bar{\boldsymbol{c}}\bar{\boldsymbol{x}}$$

with $\bar{A} =$

$$\left[\begin{array}{ccc|cc|ccc}
0 & 1 & & & & & & \\
\vdots & & \ddots & & & & & \\
0 & & 1 & & & & & \\
-\tilde{\alpha}_0 & \cdots\cdots & -\tilde{\alpha}_{r-1} & \cdots & -\alpha_{n-1} & \tilde{q}_0k^r & \cdots\cdots & \tilde{q}_{r-1}k \\
\hline
 & & & & & & & \\
 & & & \breve{A} & & & & \\
1 & & & & & & & \\
\hline
 & & & & & -p_{r-1}\kappa & 1 & \\
 & & & & & \vdots & & \ddots \\
 & & & & & -p_1\kappa^{n-1} & & 1 \\
-\tilde{\alpha}_0 & \cdots\cdots & -\tilde{\alpha}_{r-1} & \cdots & -\alpha_{r-1} & -p_0\kappa^r+\tilde{q}_0k^r & \cdots\cdots & \tilde{q}_{r-1}k
\end{array}\right]$$

$$\bar{B} = \boldsymbol{e}_r\left[\alpha_0,\ \ldots,\ \alpha_{r-1},\ 1\right]$$

$$\boldsymbol{y}_{ref} = \begin{bmatrix} y_{ref} \\ \dot{y}_{ref} \\ \vdots \\ y_{ref}^{(r)} \end{bmatrix} \quad \text{and } \bar{\boldsymbol{c}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T$$

where

$$\tilde{\alpha}_i = \alpha_i + \tilde{q}_ik^{r-i},\ i = 0,\ldots,r-1$$
$$\tilde{q}_i = q_i \cdot g$$
$$\boldsymbol{e}_r \ \text{ the r-th unit vector}$$
$$\breve{A} \in \mathbb{R}^{(n-r)\times(n-r)} \ \text{ Hurwitz}$$

as $\breve{A}$ is block-triangular with $\tilde{A}_{22}$, $\tilde{A}_{33}$ on the diagonal. Define new coordinates $\bar{\bar{\boldsymbol{x}}}$ via a gain-dependent transformation by

$$\bar{\bar{\boldsymbol{x}}} = K^{-1}\bar{\boldsymbol{x}}$$

where

$$K = \operatorname{diag}\{K_r, I_m, K_r^2\}$$
$$K_r = \operatorname{diag}\{1, k, \ldots, k^{r-1}\}.$$

Setting $\kappa = k^2$, the differential equation of the closed-loop is

$$\dot{\bar{\bar{\boldsymbol{x}}}} = D_k\bar{\bar{A}}\bar{\bar{\boldsymbol{x}}} - \bar{\bar{B}}\boldsymbol{y}_{ref}$$

with

$$D_k = \left[kI_r\ I_m\ k^2I_r\right],\ \bar{\bar{A}} =$$

$$\left[\begin{array}{cccc|ccc|cccc}
0 & 1 & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & \\
0 & & & 1 & & & & & & & \\
-\tilde{q}_0 - \frac{\alpha_0}{k^r} & \cdots\cdots & & -\tilde{q}_{r-1} - \frac{\alpha_{r-1}}{k} & -\frac{\alpha_r}{k^r} & \cdot & -\frac{\alpha_{n-1}}{k^r} & \tilde{q}_0 & \tilde{q}_1k & \cdots & \tilde{q}_{r-1}k^{r-1} \\
\hline
 & & & & & & & & & & \\
 & & & & & \breve{A} & & & & & \\
1 & & & & & & & & & & \\
\hline
 & & & & & & & -p_{r-1} & 1 & & \\
 & & & & & & & \vdots & & \ddots & \\
 & & & & & & & -p_1 & & & 1 \\
-\frac{\tilde{q}_0}{k^r} - \frac{\alpha_0}{k^{2r}} & \cdots & & -\frac{\tilde{q}_{r-1}}{k^r} - \frac{\alpha_{r-1}}{k^{r+1}} & -\frac{\alpha_r}{k^{2r}} & \cdot & -\frac{\alpha_{n-1}}{k^{2r}} & -p_0 + \frac{\tilde{q}_0}{k^r} & \frac{\tilde{q}_1}{k^{r-1}} & \cdots & \frac{\tilde{q}_{r-1}}{k}
\end{array}\right]$$

and $\bar{\bar{B}} = k^{-(r-1)}\boldsymbol{e}_r\left[\alpha_0,\ \ldots,\ \alpha_{r-1},\ 1\right]$

By assumption, the matrices

$$\bar{\bar{A}}_{11} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\tilde{q}_0 & \cdots\cdots & & -\tilde{q}_{r-1} \end{bmatrix},\ \bar{\bar{A}}_{33} = \begin{bmatrix} -p_{r-1} & 1 & & \\ \vdots & & \ddots & \\ -p_1 & & & 1 \\ -p_0 & & & \end{bmatrix}$$

$$\text{and } \bar{\bar{A}}_{22} = \breve{A}$$

are Hurwitz. Therefore, there exist unique symmetric, positive definite solutions P_1, P_2, P_3 of the Lyapunov equations

$$\bar{\bar{A}}_{ii}^TP_i + P_i\bar{\bar{A}}_{ii} = I \quad i = 1,\ldots,3. \tag{12}$$

As a Lyapunov function candidate we choose

$$V(\bar{\bar{\boldsymbol{x}}}) = D(\bar{\bar{\boldsymbol{x}}})^2$$
$$D(\bar{\bar{\boldsymbol{x}}}) = \begin{cases} \|\bar{\bar{\boldsymbol{x}}}\|_P - \rho, & \text{if } \|\bar{\bar{\boldsymbol{x}}}\|_P \geq \rho \\ 0, & \text{if } \|\bar{\bar{\boldsymbol{x}}}\|_P \leq \rho, \end{cases}$$

where

$$\rho = \lambda\sqrt{\frac{c_1}{\|P_1^{-1}\|}}, \quad \|\bar{\bar{\boldsymbol{x}}}\|_P^2 = \bar{\bar{\boldsymbol{x}}}^TP\bar{\bar{\boldsymbol{x}}}$$
$$P = \operatorname{diag}\{c_1P_1, c_2P_2, c_3P_3\}$$
$$c_1 = k^{2-2r},\ c_2 = k^{3-3r},\ c_3 = 1,$$

and define $\bar{\bar{E}} = \bar{\bar{A}} - \mathrm{diag}\{\bar{\bar{A}}_{ii}\}$.

The derivative of V *along the trajectory of the system* (10) *is for each time interval* $[t_i, t_{i+1})$ *during which* k *is constant*

$$\frac{d}{dt}V(\bar{\bar{x}}) = 2D(\bar{\bar{x}})\frac{d}{dt}D(\bar{\bar{x}}) = 2D(\bar{\bar{x}})\frac{1}{2}\frac{\frac{d}{dt}x^T P x}{\|\bar{\bar{x}}\|_P}$$

$$= \frac{D(\bar{\bar{x}})}{\|\bar{\bar{x}}\|_P}\Big(\sum_{i=1}^{3} \bar{\bar{x}}_i^T c_i P_i([D_k]_{ii}\bar{\bar{A}}_{ii}\cdot\bar{\bar{x}}_i + \bar{\bar{B}}_i y_{ref}) + \sum_{i=1}^{3}\sum_{j=1}^{3}\bar{\bar{x}}_i^T c_i P_i [D_k]_{jj}\bar{\bar{E}}_{ij}\cdot\bar{\bar{x}}_j\Big).$$

Using (12) *we have*

$$\frac{d}{dt}V(\bar{\bar{x}}) \le \frac{D(\bar{\bar{x}})}{\|\bar{\bar{x}}\|_P}\Big(-[c_1 k,\, c_2,\, c_3 k^2]\begin{bmatrix}\|\bar{\bar{x}}_1\|^2\\ \|\bar{\bar{x}}_2\|^2\\ \|\bar{\bar{x}}_3\|^2\end{bmatrix} + \sum_{i=1}^{3}\sum_{j=1}^{3} c_i\|P_i\|\,\|[D_k]_{jj}\|\,\|\bar{\bar{E}}_{ij}\|\,\|\bar{\bar{x}}_i\|\,\|\bar{\bar{x}}_j\| + \sum_{i=1}^{3} c_i\,\|P_i\|\,\|\bar{\bar{B}}_i\|\,\|y_{ref}\|\,\|\bar{\bar{x}}_i\|\Big).$$

Using that $k \ge k_0$ *and* $y_{ref} \in W^{r,\infty}$ *there exist an* $M > 0$ *s.t.*

$$\begin{aligned}\frac{d}{dt}V(\bar{\bar{x}}) \le -\frac{D(\bar{\bar{x}})}{\|\bar{\bar{x}}\|_P}\Big(& c_1(k-M)\|\bar{\bar{x}}_1\|^2 \\ &+ c_2\|\bar{\bar{x}}_2\|^2 + c_3(k^2 - kM)\|\bar{\bar{x}}_3\|^2 \\ &- M(c_1 k^{1-r} + c_2)\,\|\bar{\bar{x}}_1\|\,\|\bar{\bar{x}}_2\| \\ &- M(c_1 k + c_3 k^{2-r})\,\|\bar{\bar{x}}_1\|\,\|\bar{\bar{x}}_3\| \\ &- c_3 M k^{2-2r}\,\|\bar{\bar{x}}_2\|\,\|\bar{\bar{x}}_3\| \\ &- c_1 M\|\bar{\bar{x}}_1\| - c_2 M\|\bar{\bar{x}}_2\| - c_3 M\|\bar{\bar{x}}_3\|\Big).\end{aligned}$$

We assume that k *tends to infinity and show now that this leads to a contradiction. By monotonicity of* k*, there exists a time* $t^1 \ge 0$ *such that the gain* $k(t)$ *is sufficiently large for all* $t \ge t^1$ *to ensure that:*

$$\begin{aligned}\frac{d}{dt}V(\bar{\bar{x}}) \le -\frac{D(\bar{\bar{x}})}{\|\bar{\bar{x}}\|_P}\Big(&\frac{1}{2}k^{3-2r}\|\bar{\bar{x}}_1\|^2 + k^{3-3r}\|\bar{\bar{x}}_2\|^2 + \frac{1}{2}k^2\|\bar{\bar{x}}_3\|^2 \\ &- Mk^{3-3r}(2\,\|\bar{\bar{x}}_1\|\,\|\bar{\bar{x}}_2\| + k^{\frac{r}{2}}(\|\bar{\bar{x}}_1\| - k^{-\frac{r}{2}}\|\bar{\bar{x}}_2\|)^2) \\ &- 2Mk^{2-r}\|\bar{\bar{x}}_1\|\,\|\bar{\bar{x}}_3\| - Mk^{\frac{3}{2}}(k^{\frac{1}{2}-r}\|\bar{\bar{x}}_1\| - \|\bar{\bar{x}}_3\|)^2 \\ &- Mk^{2r-2}(\|\bar{\bar{x}}_2\|\,\|\bar{\bar{x}}_3\| - \frac{1}{2}(\|\bar{\bar{x}}_2\|^2 - \|\bar{\bar{x}}_3\|)^2) \\ &- k^{2-2r}M\|\bar{\bar{x}}_1\| - \frac{1}{2}k^{\frac{5}{2}-2r}(\|\bar{\bar{x}}_1\| - Mk^{-\frac{1}{2}})^2 \\ &- k^{3-3r}M\|\bar{\bar{x}}_2\| - \frac{1}{2}k^{3-3r}(\|\bar{\bar{x}}_2\| - M)^2 \\ &- M\|\bar{\bar{x}}_3\| - \frac{1}{2}(k^{\frac{1}{2}}\|\bar{\bar{x}}_3\| - Mk^{-\frac{1}{2}})^2\Big).\end{aligned}$$

Again by monotonicity of k*, there exists a* $t^2 \ge t^1$ *such that for all* $t \ge t^2$ *the gain* $k(t)$ *is sufficiently large to ensure that:*

$$\frac{d}{dt}V(\bar{\bar{x}}) \le -\frac{D(\bar{\bar{x}})}{3\|\bar{\bar{x}}\|_P}\Big(k^{3-2r}\|\bar{\bar{x}}_1\|^2 + k^{3-3r}\|\bar{\bar{x}}_2\|^2 + k^2\|\bar{\bar{x}}_3\|^2 - \frac{3}{2}M^2(k^{\frac{3}{2}-2r} + k^{3-3r} + k^{-1})\Big).$$

For $r > 1$*, the last term is approaching zero as* $k \to \infty$*. Thus, there exist a* $t^3 \ge t^2$ *such that for all* $t \ge t^3$ *the inequalities* $k(t) \ge 1$ *and*

$$M^2(k^{3/2-2r} + k^{3-3r} + k^{-1}) \le 3\rho^2 m, \quad \text{with} \quad m = (3\max_i \|P_i\|)^{-1}$$

are fulfilled. Therefore, for all $t \ge t^3$

$$\begin{aligned}\frac{d}{dt}V(\bar{\bar{x}}) &\le -\frac{D(\bar{\bar{x}})}{3\|x\|_P}\Big(\sum_{i=1}^{3}\frac{c_i\|P_i\|\,\|\bar{\bar{x}}_i\|^2}{\|P_i\|} - 3\rho^2 m\Big) \\ &\le -\frac{D(\bar{\bar{x}})}{3\|x\|_P}\,(3m\|\bar{\bar{x}}\|_P^2 - 3\rho^2 m) \\ &\le -mD(\bar{\bar{x}})\left(\|\bar{\bar{x}}\|_P - \rho\frac{\rho}{\|x\|_P}\right).\end{aligned}$$

As $\|x\|_P > \rho$ *whenever* $D(\bar{\bar{x}}) > 0$ *it follows that on each interval* $[t_{i^*}, t_{i^*+1})$ *with* $t_{i^*} \ge t^3$

$$\frac{d}{dt}V(\bar{\bar{x}}) \le -mD(\bar{\bar{x}})^2 = -mV(\bar{\bar{x}}). \tag{13}$$

If $|e| > \lambda$*, then with* $\epsilon = \sqrt{c_1^{-1}\|P_1^{-1}\|}$ *it holds that*

$$\begin{aligned}\dot\varphi(t) &\le (|e| - \lambda)^2 \le (\|\bar{\bar{x}}_1\| - \lambda)^2 \le (\epsilon\|\bar{\bar{x}}\|_P - \lambda)^2 \\ &\le \epsilon^2(\|\bar{\bar{x}}\|_P - \epsilon^{-1}\lambda)^2 \le \epsilon^2(\|\bar{\bar{x}}\|_P - \rho)^2 \le \epsilon^2 D(\bar{\bar{x}})^2.\end{aligned}$$

If $|e| \le \lambda$*, then* $\dot\varphi(t) = 0$ *and hence*

$$\dot\varphi(t) \le c_1^{-1}\|P_1^{-1}\|V(\bar{\bar{x}}) \tag{14}$$

holds for all $t \ge 0$.

At a switching point t_i *it holds that*

$$V(\bar{\bar{x}}(t_{i+1})) \le \mu V(\bar{\bar{x}}(t_i)),\ \mu = \mu(P) \tag{15}$$

where $\mu(\cdot)$ *denotes the condition number.*

Integration of (14) *over the time intervals during which* k *is constant yields by* (13)*, for all* $j \ge i^*$,

$$\frac{\varphi(t_{j+1}) - \varphi(t_j)}{\|P_1^{-1}\|} \le \int_{t_j}^{t_{j+1}}\frac{V(t)}{c_1(t)}dt \le \frac{V(t_j)}{c_1(t_j)m} \tag{16}$$

Now an application of (15) *to* (16) *yields for all* $i \ge 0$,

$$\varphi(t_{i^*+i+1}) - \varphi(t_{i^*}) \le \frac{\|P_1^{-1}\|\,V(t_{i^*})\,\mu^i}{m\,c_1(t_{i^*+i})}. \tag{17}$$

By the growth condition (7d) *on* φ*, there exists a finite index* $\bar\imath$*, such that the left side of* (17) *is greater than the right side. Thus, the assumption that* $\varphi(t) \to \infty$ *does not hold and therefore, the adaptation gains must be bounded:*

$$\lim_{t\to\infty} k(t) = \varphi(t_{i^*+\bar\imath}) \le \lim_{t\to\infty}\varphi(t) < \infty.$$

Setting $t^* = t_{i^*+\bar\imath}$ *completes the proof of* $\varphi \in L_\infty$ *and hence b) follows from* (7).

In the sequel consider (10) *only for* $t \ge t^*$*, so that* $t \mapsto k(t)$ *is constant.*

3) Boundedness of the observer states. *With*

$$\psi(t) := |e(t)| - d_\lambda(e(t)) \in L_\infty(0,\infty)$$

and $d_\lambda(e(\cdot)) \in L_2(0,\infty)$ *it follows that*

$$e(\cdot) = \underbrace{\psi(\cdot)\,\mathrm{sign}(e(\cdot))}_{\in L_\infty} + \underbrace{d_\lambda(e(\cdot))\,\mathrm{sign}(e(\cdot))}_{\in L_2}. \quad (18)$$

By applying (18) *to* (10b), *it follows from the theory of ordinary differential equations that* $\hat{\boldsymbol{x}}(\cdot) \in L_\infty(0,\infty)$ *(note that* $\hat{A}_{k(t)} = \hat{A}_{k(t^*)}$ *is Hurwitz).*

4) Boundedness of the states of the plant. *Since* $(\boldsymbol{c}, A)$ *is detectable, there exists some* $\boldsymbol{k} \in \mathbb{C}^{n\times 1}$ *such that* $\sigma(A - \boldsymbol{kc}) \subset \mathbb{C}^-$. *Now consider the new observer*

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}} &= A\tilde{\boldsymbol{x}} - \boldsymbol{b}\boldsymbol{q}_k\hat{\boldsymbol{x}} + \boldsymbol{kc}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \\ &= (A - \boldsymbol{kc})\tilde{\boldsymbol{x}} - \boldsymbol{b}\boldsymbol{q}_k\hat{\boldsymbol{x}} + \boldsymbol{k}(e - y_{ref}).\end{aligned}$$

Since $\boldsymbol{q}_k\hat{\boldsymbol{x}} \in L_\infty$, $y_{ref} \in L_\infty$ *and* e *is the sum of an* L_2*- and an* L_∞*-function, exponential stability of* $(A - \boldsymbol{kc})$ *yields* $\tilde{\boldsymbol{x}} \in L_\infty$. *Exploiting exponential stability of*

$$\frac{d}{dt}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) = (A - \boldsymbol{kc})(\boldsymbol{x} - \tilde{\boldsymbol{x}})$$

and boundedness of $\tilde{\boldsymbol{x}}$ *yields* $\boldsymbol{x}(\cdot) \in L_\infty(0,\infty)$.

5) Convergence of the tracking error. *It remains to prove c). For this we prove that* $\lim_{t\to\infty} d_\lambda(e(t)) = 0$. *Since* $y(\cdot)$ *is bounded it follows that* $e(\cdot)$, $d_\lambda(e(\cdot))^2 \in L_infty(0,\infty)$, *and from* $\dot{e} = \boldsymbol{c}[A\boldsymbol{x} - \boldsymbol{b}\boldsymbol{q}_k\hat{\boldsymbol{x}}] - \dot{y}_{ref}$ *and previous results we conclude* $\dot{e}(\cdot) \in L_\infty(0,\infty)$. *Now*

$$\frac{d}{dt}d_\lambda(e(t))^2 = 2d_\lambda(e(t))\frac{e(t)\dot{e}(\cdot)}{|e(t)|} \in L_\infty(0,\infty)$$

and hence $d_\lambda(e(\cdot))^2$ *is uniformly continuous. This, together with* $d_\lambda(e(\cdot))^2 \in L_1(0,\infty)$ *yields, by Barbălat's Lemma (see (Barbălat, 1959) or (Khalil, 1996)) that* $\lim_{t\to\infty} d_\lambda(e(t))^2 = 0$ *and thus c) holds.*

This completes the proof. ∎

4. CONCLUSIONS

In this paper a high-gain adaptive controller for linear, minimum-phase systems of arbitrary, but known relative degree is presented. This is the first step in the direction of λ-tracking of high relative degree nonlinear systems. The assumption of knowledge of a bound on the high frequency gain should be removed in the future and also the gain should be chosen continuously, see (Bullinger and Allgöwer, 1998). Future research will focus on nonlinear systems.

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